positions of the source: $a = 1.5, 2, 3$, and 9. As follows from physical considerations, at the initial instant of time the pressure at the leading point is doubled.

The dashed curves *1-3* in Fig.3 represent the pressure distribution over the surface of the parabolic cylinder at times $t = 0.6, 1.2$, and 3. The source of the cylindrical wave is in the plane of symmetry and $a = 2$. The calculations were carried out using four terms of the expansion (1.27).

REFERENCES

1. IVANOV V.I., The shortwave asymptotic form of a diffraction wave in the shadow of a parabolic cylinder. Radiotekhnika i Elektronika, 5, 3, 1960.
2. IVANOV V.I., Diffraction of short waves by a parabolic cylinder. Zh. vychisl. Mat. mat. Fiz, 2, 2, 1962.
3. FRIEDLANDER F.G., The reflection of sound pulses by convex parabolic reflectors. Proc. Camb. Phil. Soc., 37, 2, 1941.
4. CHESTER W., The reflection of a transient pulse by a parabolic cylinder and a paraboloid of revolution. Quart. J. Mech., 5, 2, 1952.
5. MORSE P.M. and FESHBACH H., Methods of Theoretical Physics, Pt. 2, McGraw-Hill, New York, 1953.
6. THAU S.A. and PAO Y.-H., Wave function expansions and perturbation method for the diffraction of elastic waves by a parabolic cylinder. J. Appl. Mech. E. Ser., 34, 4, 1967.
7. ABRAMOWITS M. and STEGUN I., (Eds.), Handbook of Mathematical Functions Nauka, Moscow, 1979.
8. TRICOMI F., Equazioni a derivate parziali. Cremonese, Rome, 1957.
9. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals, Sums, Series and Products, Nauka, Moscow, 1971.
10. DITKIN V.A. and PRUDNIKOV A.P., Handbook of Operational Calculus, Vysshaya Shkola, Moscow, 1965.

Translated by D.L.

PMM U.S.S.R., Vol.54, No.2, pp.226-231, 1990
Printed in Great Britain
0021-8928/90 $10.00+0.00

INHOMOGENEOUS ELASTIC STRUCTURES OPTIMAL IN STIFFNESS*

L.V. PETUKHOV and K.E. SOKOV

The problem of maximizing the stiffness (of minimizing the work of the external forces) of an elastic structure in which the shear modulus is the control or, in the two-dimensional case, the plate thickness /1-3/ is considered. Point-by-point and integral constraints are imposed on the control. Necessary Weierstrass-Erdmann conditions and Weierstrass conditions are obtained that enable qualitative deductions to be made about the optimal solution. These deductions do not agree with the results in /4/ in which, it is true, a problem of mathematical physics is examined.

1. Formulation of the problem. Let R^N be an N-dimensional Euclidean space of vectors $\mathbf{x} = x_i\mathbf{e}_i$, where $\mathbf{e}_i$ are the unit vectors of a Cartesian system of coordinates (here and everywhere henceforth the Latin subscripts i, j, k, l, m, n run through values from 1 to N and summation from 1 to N is assumed over the repeated subscripts i, j, k, l, m, n in the products), Ω is the projection domain in R^N, and Γ is the boundary of Ω.

We will assume that the domain Ω can be filled by an elastic inhomogeneous material

Prikl. Matem. Mekhan., 54, 2, 275-280, 1990

characterized by the tensor of the elastic constants $\theta(\mathbf{x})\mathbf{a}$, where

$$\theta(\mathbf{x}) \in L_\infty(\Omega), \ \int_\Omega \theta(\mathbf{x})\,dx = \theta^0 \operatorname{mes}\Omega, \theta^0 = \mathrm{const} \tag{1.1}$$

$$0 < \theta_- \leqslant \theta(\mathbf{x}) \leqslant \theta_+ \tag{1.2}$$

$$\mathbf{a} = a_{ijkl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l, \quad a_{ijkl} = \frac{2\mu\nu}{1-q\nu}\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{1.3}$$

in which $q = 1$ for $N = 2$ for the plane state of stress, $q = 2$ for $N = 3$ and $N = 2$ for the plane state of strain, and the shear modulus μ and Poisson's ratio ν are fixed constants.

Let us formulate the optimal design problem. Suppose we are given the vector of external forces $\mathbf{F}$ acting on the boundary Γ_F and the section of the boundary Γ_u on which the displacements of the elastic domain equal zero $(\Gamma_F \cap \Gamma_u = \varnothing)$, while the remaining part of the boundary Γ is without a load, $\theta^0, \theta_-, \theta_+, \nu, \mu$. It is required to obtain

$$\inf_\theta J(\mathbf{u}), J = \int_{\Gamma_F} F_i u_i \,d\Gamma \tag{1.4}$$

where $F_i \in L_2(\Gamma_F)$, while $\mathbf{u} = u_i\mathbf{e}_i$ is the solution of the integral identity

$$\int_\Omega \theta(\mathbf{x}) A(\mathbf{u}, \mathbf{v})\,dx - \int_{\Gamma_F} F_i v_i \,d\Gamma = 0, \forall \mathbf{v} \in V(\Omega) \tag{1.5}$$

$$V(\Omega) = \{\mathbf{v} = v_i(\mathbf{x})\mathbf{e}_i \mid v_i \in W_2^{(1)}(\Omega), v_i(\mathbf{y}) = 0, \mathbf{y} \in \Gamma_u\}$$

$$A(\mathbf{u}, \mathbf{v}) = a_{ijkl}\varepsilon_{ij}(\mathbf{u})\,\varepsilon_{kl}(\mathbf{v})$$

$$\varepsilon_{kl}(\mathbf{u}) = (\partial u_k/\partial x_l + \partial u_l/\partial x_k)/2$$

u_i are the displacements of the elastic domain, $A(\mathbf{v}, \mathbf{v})$ is the double specific potential elastic strain energy, and $W_2^{(1)}(\Omega)$ is the Sobolev space. It follows from (1.3) that the control in problem (1.4) is realized by the shear modulus $\theta(\mathbf{x})\mu$ of the material. The solution of the optimal design problem is a structure constructed from an inhomogeneous elastic material. The optimal control can be found by two methods:

1) for two-dimensional problems the elastic layer thickness can be the control in the case of a plane state of stress,

2) the optimal control obtained can be approximated by a material with piecewise-constant elastic characteristics.

According to the kind of functional being minimized the problem is analogous to that examined in /4/.

2. ***First variation.*** We will compile the expanded functional for which we append the left side of relationship (1.5) to the right side of equality (1.4) and assuming the optimal control $\theta^*(\mathbf{x})$ to be smooth, we find the first variation

$$\delta J = \int_{\Gamma_F} F_i \delta u_i \,d\Gamma + \int_\Omega [\theta^* A(\delta\mathbf{u}, \mathbf{v}) + \delta\theta A(\mathbf{u}^*, \mathbf{v})]\,dx \tag{2.1}$$

where $\delta\theta$, $\delta\mathbf{u}$ are variations of the control and the displacement vector. We set $\mathbf{v} = -\mathbf{u}^*$. Then we obtain the inequality

$$\int_\Omega \delta\theta A(\mathbf{u}^*, \mathbf{u}^*)\,dx \leqslant 0 \tag{2.2}$$

for $\delta\theta(\mathbf{x})$ satisfying the condition

$$\int_\Omega \delta\theta(\mathbf{x})\,dx = 0 \tag{2.3}$$

from (1.5) and the necessary condition $\delta J \geqslant 0$.

The existence of a non-negative constant ζ^*, such that

$$A(\mathbf{u}^*, \mathbf{u}^*) = \zeta^*, \ \forall \mathbf{x} \in \Omega_1$$
$$A(\mathbf{u}^*, \mathbf{u}^*) \leqslant \zeta^*, \ \forall \mathbf{x} \in \Omega_2; \ A(\mathbf{u}^*, \mathbf{u}^*) \geqslant \zeta^*, \ \forall \mathbf{x} \in \Omega_3 \tag{2.4}$$

$$\Omega_1 = \{\mathbf{x} \in \Omega \mid \theta_- < \theta^*(\mathbf{x}) \leqslant \theta_+\}$$
$$\Omega_2 = \{\mathbf{x} \in \Omega \mid \theta^*(\mathbf{x}) = \theta_-\},\ \Omega_3 = \{\mathbf{x} \in \Omega \mid \theta^*(\mathbf{x}) = \theta_+\}$$

follows from inequality (2.2) and equality (2.3).

We will now assume that the optimal control $\theta^*(\mathbf{x})$ is a discontinuous function that undergoes a discontinuity on passing through the smooth surface Γ_0 separating Ω into two parts Ω^- and Ω^+. We will use the notation

$$\mathbf{u}^* = \begin{cases} \mathbf{u}^-(\mathbf{x}), \mathbf{x} \in \Omega^- \\ \mathbf{u}^+(\mathbf{x}), \mathbf{x} \in \Omega^+ \end{cases} \tag{2.5}$$

The optimal solution $\mathbf{u}^*(\mathbf{x})$ remains continuous on passing through Γ_0; however the derivatives may undergo a discontinuity.

We introduce curvilinear orthogonal coordinates τ_k, on the surface Γ_0, and let τ_N be a Cartesian coordinate orthogonal to Γ_0 /5/. We will find the connection between the derivatives of $\mathbf{u}^*$ on passing through Γ_0.

The equalities

$$\mathbf{r}_k \cdot \nabla \mathbf{u}^- |_{\Gamma_0} = \mathbf{r}_k \cdot \nabla \mathbf{u}^+ |_{\Gamma_0}, \quad k = 1, \ldots, N-1 \tag{2.6}$$
$$\mathbf{r}_N \cdot \sigma(\mathbf{u}^-) |_{\Gamma_0} = \mathbf{r}_N \cdot \sigma(\mathbf{u}^+) |_{\Gamma_0} \quad (\sigma = \theta \mathbf{a} \cdot\cdot\, \varepsilon(\mathbf{u}))$$

follow from (2.1) and (2.5), where σ is the stress tensor computed for the field of the displacements $\mathbf{u}$, and $\mathbf{r}_k$ are unit vectors associated with the curvilinear coordinates τ_k introduced (here and everywhere later the scalar and double scalar products are denoted by single and double dots /6/). We will consider $\mathbf{u}$, $\varepsilon(\mathbf{u})$, $\sigma(\mathbf{u})$ referred to the coordinates τ_k. Then the last equality in (2.6) can be written in the form

$$\theta^-(\mathbf{r}_N \cdot \mathbf{a} \cdot \mathbf{r}_k) \cdot (\mathbf{r}_k \cdot \nabla \mathbf{u}^-) = \theta^+(\mathbf{r}_N \cdot \mathbf{a} \cdot \mathbf{r}_k) \cdot (\mathbf{r}_k \cdot \nabla \mathbf{u}^+)$$

from which we obtain an expression for $\mathbf{r}_N \cdot \nabla \mathbf{u}^-$ by taking account of the first two equalities (2.6), and we find the jump $A(\mathbf{u}^*, \mathbf{u}^*)$ for the passage through Γ_0

$$A(\mathbf{u}^-, \mathbf{u}^-) = \nabla \mathbf{u}^- \cdot\cdot\, \mathbf{a} \cdot\cdot\, \nabla \mathbf{u}^- = A(\mathbf{u}^+, \mathbf{u}^+) - (\theta^- - \theta^+)(\theta^- + \theta^+)(\theta^-\theta^+)^{-2} X(\mathbf{r}_N) \tag{2.7}$$

$$X(\mathbf{r}_N) = \begin{cases} [\sigma_{31}^2 + \sigma_{32}^2 + \frac{1}{2}(1 - 2\nu)\sigma_{33}^2]\mu^{-1}, & N = 3 \\ [\sigma_{21}^2 + \frac{1}{2}(1 - q\nu)\sigma_{22}^2]\mu^{-1}, & N = 2 \end{cases}$$

(the relationship (1.3) is taken into account in the expression for $X(\mathbf{r}_N)$). The stress tensor components are here represented in the coordinates τ_k.

Analysis of relationships (2.7) shows that since $X(\mathbf{r}_N) \geqslant 0$, we have

$$A(\mathbf{u}^-, \mathbf{u}^-)|_{\Gamma_0} \leqslant A(\mathbf{u}^+, \mathbf{u}^+)|_{\Gamma_0}, \theta^-|_{\Gamma_0} > \theta^+|_{\Gamma_0} \tag{2.8}$$

On the other hand, the inequality

$$A(\mathbf{u}^-, \mathbf{u}^-)|_{\Gamma_0} \geqslant A(\mathbf{u}^+, \mathbf{u}^+)|_{\Gamma_0}, \theta^-|_{\Gamma_0} > \theta^+|_{\Gamma_0} \tag{2.9}$$

follows from the necessary conditions (2.4).

Comparing (2.8) and (2.9), we obtain that the jump in the control θ on the smooth surface Γ_0 is possible only in the case when

$$\mathbf{r}_N \cdot \sigma(\mathbf{u})|_{\Gamma_0} = 0, A(\mathbf{u}^-, \mathbf{u}^-)|_{\Gamma_0} = A(\mathbf{u}^+, \mathbf{u}^+)|_{\Gamma_0} = \zeta^* \tag{2.10}$$

on this surface.

3. ***Weierstrass's necessary condition.*** To obtain Weierstrass's necessary condition at the point $\mathbf{x}_0 \in \Omega$ we consider the simply-connected domain Ω_0 that is stellar in $\mathbf{x}_0$, where $\bar{\Omega}_0 \in \Omega$. We take the point $\mathbf{y} \in \Gamma_0$ (Γ_0 is the boundary of Ω_0) and we draw a vector $\mathbf{r}(\mathbf{y})$ to it from the point $\mathbf{x}_0$. If the set of points $\eta\mathbf{r}(\mathbf{y})$, is considered, then a boundary $\Gamma_0(\eta)$ is obtained that extracts the domain $\Omega_0(\eta)$, where $\Omega_0 = \Omega_0(1)$, $\Gamma_0 = \Gamma_0(1)$. The domain $\Omega_0(\eta)$ is obtained from Ω_0 by an η-fold change of all its linear dimensions, consequently

$$\operatorname{mes} \Omega_0(\eta) = \eta^N \operatorname{mes} \Omega_0 \tag{3.1}$$

We will assume that $\theta^*(\mathbf{x})$ is a piecewise-continuous optimal control, each continuous part of which is smooth. We take the point $\mathbf{x}_0 \in \Omega$, at which the continuity of $\theta^*(\mathbf{x})$ is not disturbed. We construct a domain $\Omega_0(\eta)$, $0 \leqslant \eta < \eta_0 < 1$, for $\mathbf{x}_0$ such that the function $\theta^*(\mathbf{x})$ is smooth therein. We give an arbitrary control θ satisfying the inequalities (1.2) in $\Omega_0(\eta)$ and we give $\theta(\mathbf{x}, \eta)$ in the domain $\Omega \setminus \Omega_0(\eta)$ such that $\theta(\mathbf{x}, 0) = \theta^*(\mathbf{x})$ and

$$\int_{\Omega\setminus\Omega_0(\eta)} \Delta\theta\, dx + \int_{\Omega_0(\eta)} \theta\, dx = \theta^0 \operatorname{mes}\Omega \tag{3.2}$$
$$\Delta\theta = \theta_0 - \theta^*,\ \theta(\mathbf{x}, 0) = \theta^*(\mathbf{x}),\ \mathbf{x} \in \Omega_0(\eta)$$

It follows from (3.1) and (3.2)

$$\delta\theta = \ldots = \delta^{N-1}\theta = 0,\ \delta^N\theta = 0 \tag{3.3}$$
$$\Delta\theta(\mathbf{x}_0) N! \operatorname{mes}\Omega_0 + \int_{\Omega\setminus\Omega_0(\eta)} \delta^N\theta\, dx = 0 \tag{3.4}$$

We set up the extended functional

$$J = J_1 + J_2,\quad J_1 = -\int_{\Omega\setminus\Omega_0(\eta)} \Delta\theta A(\mathbf{u}, \mathbf{u}^*)\, dx$$
$$J_2 = -\int_{\Omega_0(\eta)} \theta A(\mathbf{u}, \mathbf{u}^*)\, dx + \int_{\Gamma_F} F_i (u_i + u_i^*)\, d\Gamma$$

It follows from (3.3) that

$$\delta J_2 = \ldots = \delta^{N-1} J_2 = 0,\quad \delta^N J_2 = -\int_\Omega \delta^N\theta A(\mathbf{u}^*, \mathbf{u}^*)\, dx$$

The function $\mathbf{u}^*$ is differentiable in the domain $\Omega_0(\eta)$, and consequently, by applying the formula

$$\Delta\theta A(\mathbf{u}, \mathbf{u}^*) = \nabla\cdot\left(\frac{\Delta\theta}{\theta^*}\sigma(\mathbf{u}^*)\cdot\mathbf{u}\right) - \nabla\frac{\Delta\theta}{\theta^*}\sigma(\mathbf{u}^*)\cdot\mathbf{u} - \frac{\Delta\theta}{\theta^*}\nabla\cdot\sigma(\mathbf{u}^*)\cdot\mathbf{u}$$

to the integral J_1 and using Ostrogradskii's formula, we obtain

$$J_1 = -\int_{\Gamma_0(\eta)} \frac{\Delta\theta}{\theta^*}\mathbf{r}\cdot\sigma(\mathbf{u}^*)\cdot\mathbf{u}\, d\Gamma + \int_{\Omega_0(\eta)} \nabla\frac{\Delta\theta}{\theta^*}\cdot\sigma(\mathbf{u}^*)\cdot\mathbf{u}\, dx \tag{3.5}$$

where $\mathbf{r}$ is the external unit normal to the boundary $\Gamma_0(\eta)$ of the domain $\Omega_0(\eta)$ (it is also taken into account that $\nabla\cdot\sigma(\mathbf{u}^*) = 0,\ \mathbf{x} \in \Omega_0(\eta)$).

In the domain $\Omega_0(\eta)\ \mathbf{u}(\mathbf{x}, \eta) \sim \eta$, and consequently, the first integral on the right-hand side of (3.5) is proportional to η^N and the second to η^{N+1}. Multiplying (3.4) by ζ^* and combining with $\delta^N J = \delta^N J_1 + \delta^N J_2 \geqslant 0$, we find the inequality

$$-\frac{d^N}{d\eta^N}\left[\int_{\Gamma^0(\eta)} \frac{\Delta\theta}{\theta^*}\mathbf{r}\cdot\sigma(\mathbf{u}^*)\cdot\mathbf{u}\, d\Gamma\right]_{\eta=0} \geqslant -\Delta\theta\zeta^* N! \operatorname{mes}\Omega_0 \tag{3.6}$$

that is the necessary Weierstrass condition of a strong minimum.

In order to use (3.6), it is necessary to have the solution $\mathbf{u}(\mathbf{x}, \eta)$. It is not possible to find it for arbitrary $\Omega_0(\eta)$, however, this solution can be found for elliptic, hypotrochoidal, and ellipsoidal inclusions as $\eta \to 0$.

4. The necessary Weierstrass condition for an ellipse **($N = 2$).** Let Ω_0 be an elliptic inclusion with semimajor and semiminor axes $\eta(1 + \xi)$ and $\eta(1 - \xi),\ 0 \leqslant \xi \leqslant 1$, whose centre is at the point $\mathbf{x}_0$. We will consider the principal stress $\sigma_1 = \sigma_1(\mathbf{u}^*(\mathbf{x}_0))$ of the tensor $\sigma = \sigma(\mathbf{u}^*(\mathbf{x}_0))$ to act at an angle β to the major semi-axis of the ellipse. The solution $\mathbf{u}(\mathbf{x}, \eta)$, on the left-hand side of condition (3.6) is identical with the solution of the compression-tension problem of an infinite plane with an elliptic inclusion by forces $\sigma_1(\mathbf{u}^*(\mathbf{x}_0))$, $\sigma_2(\mathbf{u}^*(\mathbf{x}_0))$ acting at an angle β to the semimajor and semiminor axes of the ellipse at infinity as $\eta \to 0$ and is determined by the Kolosov-Muskhelishvili formula /7/

$$\begin{aligned}\mathbf{u} = {}^1/_8\eta(\varkappa + 1)\mu^{-1}\{[(1 + \xi)(\varkappa A_1 - A_1 - 2B_1)\cos\varphi - (1 - {}\\ \xi)(\varkappa A_2 + A_2 - 2B_2)\sin\varphi]\mathbf{e}_1 + [(1 + \xi)(\varkappa A_2 + A_2 + {}\\ 2B_2)\cos\varphi - (1 - \xi)(\varkappa A_1 - A_1 + 2B_1)\sin\varphi]\,\mathbf{e}_2\end{aligned} \tag{4.1}$$
$$\begin{aligned}A_1 = \{(\sigma_1 + \sigma_2)[(\varkappa + 1)\theta^* + (1 - \xi^2)\Delta\theta] + 2(\sigma_1 - {}\\ \sigma_2)\xi\Delta\theta\cos 2\beta\}R^{-1}\end{aligned}$$

$$A_2 = -\frac{2(\sigma_1 - \sigma_2)\Delta\theta\xi \sin 2\beta}{(\varkappa+1)\theta^*[\theta^* + \varkappa(\theta^* + \Delta\theta) + \Delta\theta\xi^2]}$$
$$B_1 = -\{(\sigma_1 + \sigma_2)(\varkappa - 1)\xi\Delta\theta + (\sigma_1 - \sigma_2)[(\varkappa + 1)\theta^* + 2\Delta\theta]\cos 2\beta\} R^{-1}$$
$$B_2 = (\sigma_1 - \sigma_2)[(\varkappa + 1)\theta^* + (\varkappa + \xi^2)\Delta\theta]^{-1}\sin 2\beta$$

where φ is the angle measured from the x_1 axis $\varkappa = 3 - 4\nu$ for the plane state of strain and $\varkappa = (3 - \nu)(1 + \nu)^{-1}$ for the plane state of stress.

Substituting the function $\mathbf{u}^*$, expression (4.1) and $\mathbf{r} = [(1 - \xi)\cos\varphi \mathbf{e}_1 + (1 + \xi)\sin\varphi \mathbf{e}_2]/Q$, $d\Gamma = \eta Q d\varphi$, $Q = \sqrt{1 - 2\xi\cos 2\varphi + \xi^2}$ in the left-hand side of condition (3.6), and after reduction we obtain

$$\Delta\theta(1 - \xi^2)[{}^1/_8(\varkappa + 1)(\mu\theta^*)^{-1}\Psi(\beta, \xi, \Delta\theta) + \zeta^*] \geqslant 0 \tag{4.2}$$
$$\psi(\beta, \xi, \Delta\theta) = \{(\sigma_1 + \sigma_2)^2(\varkappa - 1)\Delta\theta\xi^2 - 4(\sigma_1^2 - \sigma_2^2)(\varkappa - 1)\xi\Delta\theta\cos 2\beta - 2(\sigma_1 - \sigma_2)^2[(\varkappa + 1)\theta^* + 2\Delta\theta]\cos^2 2\beta + (\sigma_1 + \sigma_2)^2(\varkappa - 1)[(\varkappa + 1)\theta^* + \varkappa\Delta\theta]\} R^{-1} - 2(\sigma_1 - \sigma_2)^2\Delta\theta\sin 2\beta[(\varkappa + 1)\theta^* + (\varkappa + \xi^2)\Delta\theta]^{-1}$$
$$R = [(\varkappa + 1)\theta^*]^2 + (\varkappa + 1)(\varkappa + 2 - \xi^2)\theta^*\Delta\theta + 2\varkappa(1 - \xi^2)(\Delta\theta)^2$$

(for the ellipse $\operatorname{mes}\Omega_0 = \pi(1 - \xi^2)$).

Inequality (4.2) should be satisfied for any $\beta \in [0, 2\pi]$ and consequently, by setting $\sigma_1^2 \geqslant \sigma_2^2$, to be more specific, and solving the problem of minimizing the left-hand side of inequality (4.2) for $0 \leqslant \beta \leqslant \pi$, we obtain

$$\beta_* = 0, \quad \Psi_*(\xi, \Delta\theta) = \Psi(0, \xi, \Delta\theta) = \{(\sigma_1 + \sigma_2)^2(\varkappa - 1)[(\varkappa + 1)\theta^* + (\varkappa - \xi^2)\Delta\theta] + 4(\sigma_1^2 - \sigma_2^2)(\varkappa - 1)\xi\Delta\theta + 2(\sigma_1 - \sigma_2)^2[(\varkappa + 1)\theta^* + 2\Delta\theta]\} R^{-1}$$

Appending the component $A(\mathbf{u}^*, \mathbf{u}^*) - A(\mathbf{u}^*, \mathbf{u}^*)$ to the expression in square brackets on the left-hand side of inequality (4.2), we find after reduction

$$\frac{(\Delta\theta)^2(1 - \xi^2)}{4\mu(\theta^*)^2 R}\{(\sigma_1 + \sigma_2)^2(\varkappa - 1)[(\varkappa + 1)\theta^* + \varkappa(1 - \xi^2)\Delta\theta] - 2(\sigma_1^2 - \sigma_2^2)(\varkappa^2 - 1)\xi\theta^* + (\sigma_1 - \sigma_2)^2[(\varkappa + 1)(\varkappa - \xi^2)\theta^* + 2\varkappa(1 - \xi^2)\Delta\theta]\} + \Lambda \geqslant 0, \quad \Lambda = (1 - \xi^2)\Delta\theta[\zeta^* - A(\mathbf{u}^*, \mathbf{u}^*)] \tag{4.3}$$

It follows from the necessary conditions (2.4) that $\Lambda \geqslant 0$, the factor in front of the braces in (4.3) is also non-negative, and consequently the expression in the braces will be negative for

$$\Delta\theta < f(\xi), \quad f(\xi) = -\varkappa^{-1}(1 - \xi^2)^{-1}[\tau^2(\varkappa - 1) - 2\tau(\varkappa - 1)\xi + (\varkappa - \xi^2)]\Theta$$
$$\Theta = (\varkappa + 1)[\tau^2(\varkappa - 1) + 2]^{-1}\theta^*, \quad \tau = (\sigma_1 + \sigma_2)/(\sigma_1 - \sigma_2)$$

Maximizing $f(\xi)$ in the segment $0 \leqslant \xi \leqslant 1$, we find

$$f_* = f(\xi^*) = \begin{cases} -(\varkappa - 1)^{-1}\Theta, \ \xi^* = \tau, \ -1 \leqslant \sigma_2/\sigma_1 \leqslant 0 \\ -\varkappa^{-1}[\tau^2(\varkappa - 1) + 1]\Theta, \ \xi^* = \tau^{-1}, \ 0 \leqslant \sigma_2/\sigma_1 \leqslant 1 \end{cases} \tag{4.4}$$

The same relationship between the axes of the ellipse was obtained in /8/. Analysis shows that, for all possible values of σ_1, σ_2 and ν, negative values of the expression in the braces in inequality (4.3) are possible for

$$\Delta\theta < f_* < -\theta^* \tag{4.5}$$

Taking (1.2) into account we obtain

$$\theta_- - \theta^* \leqslant \Delta\theta \leqslant \theta_+ - \theta^*$$

from which and from (4.5) it follows that the expression in the braces in (4.3) is non-negative for any allowable $\Delta\theta$ and any σ_1, σ_2. Therefore, the necessary Weierstrass condition for an elliptic inclusion is always satisfied.

The Weierstrass condition may be violated in the problem of a minimum of the electrical resistance of a plane domain /4/, where the worst case of an inclusion is an ellipse degenerating into a slot. An analogous deduction about the impossibility of sliding modes /9/ holds in the problem of maximizing the plate stiffness.

In conclusion, we note that the Weierstrass-Erdmann condition for the stiffness minimization problem will be satisfied on discontinuities of $\theta^*(\mathbf{x})$ while the Weierstrass condition will not be satified at points $\mathbf{x}$ in which $\theta_- \leqslant \theta^*(\mathbf{x}) \leqslant \theta_+$.

REFERENCES

1. PRAGER W., Principles of the Theory of Optimal Design, Mir, Moscow, 1977.
2. BANICHUK N.V., Introduction to Structure Optimization. Nauka, Moscow, 1986.
3. PETUKHOV L.V. and REPIN S.I., Application of the method of duality in optimization problems for elastic body shape, PMM, 48, 5, 1984.
4. LUR'E K.A., Optimal Control in Problems of Mathematical Physics. Nauka, Moscow, 1975.
5. PETUKHOV L.V., On optimal problems of elasticity theory with unknown boundaries, PMM, 50, 2, 1986.
6. LUR'E A.I., Theory of Elasticity. Nauka, Moscow, 1970.
7. HARDIMAN N.J., Elliptic elastic inclusion in an infinite elastic plate. Quart. J. Mech. Appl. Math., 7, 2, 1954.
8. PETUKHOV L.V., Optimal elastic domains of maximal stiffness, PMM, 53, 1, 1989.
9. DIDENKO N.I. and SAMSONOV A.I., On optimization of elastic Reissner plates and sandwich plates under complex loading, Prikl. Mekhan., 24, 7, 1988.

Translated by M.D.F.

PMM U.S.S.R.,Vol.54,No.2,pp.231-242,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

ON THE STATE OF STRESS AND STRAIN NEAR CONE APICES*

N.V. MOVCHAN and S.A. NAZAROV

The asymptotic form of the state of stress and strain near the apices of inclusions or cavities having the form of a pointed cone is investigated. An arbitrary simple closed contour in a plane bounding a set g_ε of a small parameter ε is the directrix of the conical surface. The principal term of the asymptotic form $\varepsilon^2\Lambda_2 + O(\varepsilon^3)$ of the stress singularity index is calculated and examples are considered. The problem of the axisymmetric strain of an elastic half-space with a thin conical recess is investigated.

1. A pointed conical inclusion and recess. Let k_ε denote a thin cone $\{\mathbf{x} \in \mathbf{R}^3: x_3 > 0, \varepsilon^{-1}x_3^{-1}\mathbf{x}' \in g, \mathbf{x}' = (x_1, x_2)\}$, where ε is a small positive parameter, and g is a domain in the plane bounded by a simple smooth contour ∂g. We will consider the cones k_ε and $K_\varepsilon = \mathbf{R}^3 \setminus \bar{k}_\varepsilon$ filled with elastic isotropic materials with Lamé constants λ°, μ° and λ, μ, respectively, and the material contact is ideal (without peeling and slippage). It is known that the behaviour of the state of stress and strain near a conical point O is governed by the eigennumbers and vectors of a certain eigenvalue problem in the domain cut out of the cone by a unit sphere S. We introduce spherical coordinates (ρ, θ, φ), where $\rho = |\mathbf{x}|, \theta \in [0, \pi]$ is the latitude, $\varphi \in [0, 2\pi)$ is the longitude, and $\rho^{-2}Q(\theta, \varphi, \rho\partial/\partial\rho, \partial/\partial\theta, \partial/\partial\varphi)$ will denote the matrix operator of the Lamé system. We write the stress vector normal to the surface ∂K_ε in an analogous form $\rho^{-1}P(\theta, \varphi, \rho\partial/\partial\rho, \partial/\partial\theta, \partial/\partial\varphi)\mathbf{u}$. Here $\mathbf{u}$ is the displacement vector. (To abbreviate the notation, the arguments θ, φ and $\partial/\partial\theta, \partial/\partial\varphi$ will not be indicated everywhere later.). Let g_ε° be the set cut out by the cone k_ε on the sphere S. The problem with the complex spectrum parameter $\Lambda(\varepsilon)$ has the form

$$Q(\Lambda(\varepsilon))\mathbf{v} = 0 \text{ on } S \setminus g_\varepsilon^\circ \tag{1.1}$$

*Prikl.Matem.Mekhan.,54,2,281-293,1990